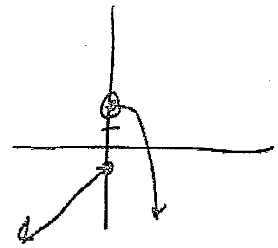
Math 250 Final Exam Review - Key

1\. a) $f(x) = \begin{cases} -x^2+2 & x > 0 \\ x-1, & x \le 0 \end{cases}$



b) i) $\lim_{x\to 1} f(x) = \lim_{x\to 1} (-x^2+2) = \boxed{1}$

ii) $\lim_{x\to 0} f(x)$ $\boxed{\text{dne}}$

iii) $f(0) = \boxed{-1}$

2\. $\lim_{x\to 3} \frac{\sqrt{(x-3)^2}}{x-3} = \lim_{x\to 3} \frac{|x-3|}{x-3} \to \boxed{\text{d.n.e.}}$

$\lim_{x\to 3^+} \frac{|x-3|}{x-3} = 1$ but $\lim_{x\to 3^-} \frac{|x-3|}{x-3} = -1$

3\. $\lim_{x\to 0} \frac{5x}{\underbrace{x^2-x}_{x(x-1)}} = \lim_{x\to 0} \frac{5}{x-1} = \boxed{-5}$

4\. $\lim_{x\to 3} \frac{(\sqrt{2}-\sqrt{x-1})}{(x-3)} \frac{(\sqrt{2}+\sqrt{x-1})}{(\sqrt{2}+\sqrt{x-1})} = \lim_{x\to 3} \frac{2-(x-1)}{(x-3)(\sqrt{2}+\sqrt{x-1})}$

$\Rightarrow \lim_{x\to 3} \frac{-x+3}{(x-3)(\sqrt{2}+\sqrt{x-1})} = \lim_{x\to 3} \frac{-1}{\sqrt{2}+\sqrt{x-1}} = \boxed{\frac{-1}{2\sqrt{2}}}$

5\. $\lim_{x\to 0} \frac{\left(\frac{2}{x+3} - \frac{2}{3}\right) 3(x+3)}{(x)\,3(x+3)} = \lim_{x\to 0} \frac{6-2(x+3)}{3x(x+3)} = \lim_{x\to 0} \frac{-2x}{3x(x+3)}$

$= \lim_{x\to 0} \frac{-2}{3(x+3)} = \boxed{-\frac{2}{9}}$

6\. $\lim_{x\to 0} \frac{\sin 3x}{2x} \cdot \frac{3}{3} = \lim_{x\to 0} \frac{\sin 3x}{3x} \cdot \frac{3}{2} = 1 \cdot \frac{3}{2} = \boxed{\frac{3}{2}}$

* $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$

7. $\lim_{x \to 2^+} \frac{5x}{x-2} \to \boxed{+\infty}$

8. $\lim_{x \to -1^-} \frac{x-3}{x+1} \to \boxed{+\infty}$

9. $\lim_{x \to \infty} \frac{5x}{x-2} = \lim_{x \to \infty} \frac{5x}{x} = \boxed{5}$

10. $\lim_{x \to \infty} \frac{3x^2-4x+1}{2-x^3} = \lim_{x \to \infty} \frac{3x^2}{-x^3} = \boxed{0}$

11. a) 1 b) 2 c) -2 d) -2

e) dne f) -2 g) 2 h) 3

12. a) Possible disc. at $x=-1$, $x=2$

b) $\left.\begin{array}{l} \lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (2) = 2 \\ \lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} (3x-1) = -4 \end{array}\right\}$ $\lim_{x \to -1} f(x)$ dne so $f(x)$ is <u>not</u> cont. at $x=-1$

$\left.\begin{array}{l} \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x) = 2 \\ \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (2) = 2 \end{array}\right\}$ $\lim_{x \to 2} f(x) = 2$

<u>Also</u> $f(2) = 2$ $\therefore \lim_{x \to 2} f(x) = f(2)$

$f(x)$ <u>is</u> <u>cont.</u> at $x=2$

<u>Ch.3</u>

1) $\boxed{f' = -10x + 4x^{-\frac{1}{2}} - \frac{20}{3}x^{-6}}$

2) $\boxed{\frac{dy}{dx} = 2\sin x + 4\cos x}$

3) $g'(t) = (4t^2+3)(2) + (2t-1)(8t)$

$g'(t) = 8t^2 + 6 + 16t^2 - 8t$

$\boxed{g'(t) = 24t^2 - 8t + 6}$

4. $P(t) = 4(5-2t)^{-\frac{1}{2}} \rightarrow P'(t) = -2(5-2t)^{-\frac{3}{2}}(-2)$

$$\boxed{P'(t) = 4(5-2t)^{-\frac{3}{2}}}$$

5. $f'(x) = x^3 \cdot \sec^2 x + \tan x \cdot 3x^2$

$$\boxed{f'(x) = x^2(x\sec^2 x + 3\tan x)}$$

6. $M'(t) = \dfrac{(7-2t)(3) - (3t-8)(-2)}{(7-2t)^2} = \dfrac{21 - 6t + 6t - 16}{(7-2t)^2}$

$$\boxed{M'(t) = \frac{5}{(7-2t)^2}}$$

7. $P'(x) = \dfrac{(1-\sin x)\cdot 1 - x(-\cos x)}{(1-\sin x)^2} = \boxed{\dfrac{1-\sin x + x\cos x}{(1-\sin x)^2}}$

8. $f'(x) = -2\csc(3x)\cot(3x)\cdot 3 - \csc^2(3x)\cdot 3$

$$\boxed{f'(x) = -6\csc(3x)\cot(3x) - 3\csc^2(3x)}$$

9. Point: $x = \pi/3 \rightarrow y = \sec \pi/3 = 2 \quad (\pi/3, 2)$

Slope: $\dfrac{dy}{dx} = \sec x \tan x \rightarrow \left.\dfrac{dy}{dx}\right|_{x=\pi/3} = 2\cdot\sqrt{3}$

$$\boxed{y - 2 = 2\sqrt{3}(x - \pi/3)}$$

10. $P'(z) = -\csc z \cot z$

$P''(z) = -[\csc z \cdot -\csc^2 z + \cot z \cdot -\csc z \cot z]$

$= \boxed{\csc^3 z + \csc z \cot^2 z}$

11. $f(t) = 5t^3 + \sqrt{t}$

a) $v(t) = f'(t) = 15t^2 + \frac{1}{2}t^{-1/2}$

$$\boxed{v(4) = 240.25 \text{ ft/sec}}$$

b) $a(t) = 30t - \frac{1}{4}t^{-3/2}$

$$\boxed{a(4) = 119.96875 \text{ ft/sec}^2}$$

12. $f(x) = 10(x^3+4)^{-1} \Rightarrow f'(x) = -10(x^3+4)^{-2}(3x^2) = \dfrac{-30x^2}{(x^3+4)^2}$

Pt: $(1, 2)$ Slope: $f'(1) = -6/5$

$$\boxed{\text{Line: } y-2 = -6/5(x-1)}$$

13. $h = f[g(x)] \rightarrow h'(x) = f'[g(x)] \cdot g'(x)$

$h'(2) = f'[g(2)] \cdot g'(2) \Rightarrow f'[5] \cdot g'(2)$

$h'(2) = 9 \cdot -3 = \boxed{-27}$

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14. $\frac{d}{dx}(-8x^2 + 5xy + y^3) = \frac{d}{dx}(-26) \Rightarrow -16x + 5x\frac{dy}{dx} + 5y + 3y^2\frac{dy}{dx} = 0$

a) $5x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 16x - 5y \Rightarrow \boxed{\dfrac{dy}{dx} = \dfrac{16x-5y}{5x+3y^2}}$

b) $\left.\dfrac{dy}{dx}\right|_{(1,-2)} = \dfrac{26}{17} \Rightarrow \boxed{y+2 = \frac{26}{17}(x-1)}$

15. $x^2 - 3y^2 = 12$

$\Rightarrow 2x - 6y\frac{dy}{dx} = 0$

$\dfrac{dy}{dx} = \dfrac{-2x}{-6y}$

$\dfrac{dy}{dx} = \dfrac{x}{3y}$

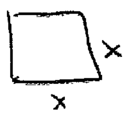
$\dfrac{d^2y}{dx^2} = \dfrac{3y \cdot 1 - x \cdot 3\frac{dy}{dx}}{(3y)^2} = \dfrac{3y - 3x\left(\frac{x}{3y}\right)}{9y^2}$

$\dfrac{d^2y}{dx^2} = \dfrac{\left(3y - \frac{x^2}{y}\right)y}{(9y^2)y} = \boxed{\dfrac{3y^2-x^2}{9y^3}}$

or $\dfrac{-12}{9y^3} = \boxed{\dfrac{-4}{3y^3}}$

16. $A = x^2$

$G: \frac{dx}{dt} = -.2 \text{ in/min.}, \; x = 4 \text{ in.}$



$\frac{dA}{dt} = 2x \frac{dx}{dt} \rightarrow \frac{dA}{dt} = 2(4)(-.2) = \boxed{-1.6 \text{ in}^2/\text{min.}}$

17.

$G: \frac{dr}{dt} = 1.2 \text{ cm/sec}, \; r = 6 \text{ cm}$

$F: \frac{dV}{dt} = ?$

$K: V = \frac{4}{3}\pi r^3 \rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \Rightarrow \frac{dV}{dt} = 4\pi(36)(1.2)$

$$\boxed{\frac{dV}{dt} = 172.8\pi \text{ cm}^3/\text{sec.}}$$

18.

$\frac{dV}{dt} = 90 \text{ m}^3/\text{hr.}, \quad dh/dt = ?, \; h = 12 \text{ m}$

$h = 3 \cdot d, \; d = 2r$

$h = 6 \cdot r$

$\frac{h}{6} = r$

$V = \frac{1}{3}\pi r^2 h \rightarrow V = \frac{1}{3}\pi \left(\frac{h}{6}\right)^2 h \rightarrow V = \frac{\pi}{108} h^3$

$\frac{dV}{dt} = \frac{\pi}{36} \cdot h^2 \frac{dh}{dt} \rightarrow 90 = \frac{\pi}{36}(12)^2 \frac{dh}{dt}$

$$\boxed{\frac{dh}{dt} = \frac{45}{2\pi} \approx 7.162 \text{ m/hr.}}$$

Ch. 4

1) $f(x) = 6x^{2/3} - 2x$

$f' = 4x^{-1/3} - 2 = 0$

$\frac{4}{\sqrt[3]{x}} = 2$

$4 = 2\sqrt[3]{x}$

$2 = \sqrt[3]{x}$

$\boxed{x = 8.}$ ← not in window

Also, $f'(x)$ dne at $\boxed{x = 0}$

x	f(x)
-1	8
0	0
1	4

$$\boxed{\begin{array}{l}\text{Abs. max} = 8 \\ \text{Abs. min} = 0\end{array}}$$

2. $f(x) = x^2 + 2x + 3, \quad [-1, 4]$

a) $f(x)$ is continuous and differentiable on $[-1, 4]$ because it is a polynomial.

b) $f'(c) = \dfrac{f(b) - f(a)}{b - a} \Rightarrow 2c + 2 = \dfrac{f(4) - f(-1)}{4 - -1} = \dfrac{27 - 2}{5} = 5$

$2c + 2 = 5 \rightarrow 2c = 3 \rightarrow \boxed{c = 3/2}$

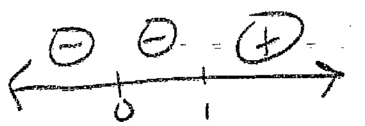
3. $f(x) = 2x^{5/3} - 5x^{4/3} \rightarrow f'(x) = \frac{10}{3}x^{2/3} - \frac{20}{3}x^{1/3}$

$f''(x) = \frac{20}{9}x^{-1/3} - \frac{20}{9}x^{-2/3} = \frac{20}{9}\left(\frac{1}{\sqrt[3]{x}} - \frac{1}{\sqrt[3]{x^2}}\right)$

$f''(x)$ is undefined at $x = 0$

$f''(x) = 0$ if $\frac{1}{\sqrt[3]{x}} - \frac{1}{\sqrt[3]{x^2}} = 0 \rightarrow \frac{1}{\sqrt[3]{x}} = \frac{1}{\sqrt[3]{x^2}}$

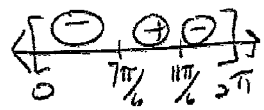
$\sqrt[3]{x^2} = \sqrt[3]{x} \rightarrow x^2 = x \rightarrow x^2 - x = 0$
$x(x-1) = 0 \quad x = 1$



$\boxed{\text{One p.o.i. at } x = 1} \Rightarrow \boxed{(1, -3)}$

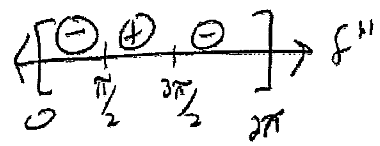
4. $f(x) = -x + 2\cos x \quad [0, 2\pi]$

a) $f'(x) = -1 - 2\sin x = 0$
$\sin x = -1/2$
$x = 7\pi/6, \ 11\pi/6$



Dec. on $[0, 7\pi/6) \cup (11\pi/6, 2\pi]$
Inc. on $(7\pi/6, 11\pi/6)$
Rel. min at $(7\pi/6, -7\pi/6 - \sqrt{3})$
Rel. max at $(11\pi/6, -11\pi/6 + \sqrt{3})$

b) $f''(x) = -2\cos x = 0$
$x = \pi/2, 3\pi/2$



c.d. on $[0, \pi/2) \cup (3\pi/2, 2\pi]$
c.u. on $(\pi/2, 3\pi/2)$
P.O.I.: $(\pi/2, -\pi/2)$
$(3\pi/2, -3\pi/2)$

5.

Primary: $A = xy$ Secondary: $x + 2y = 600$

$x = 600 - 2y$

$A = (600-2y)y$

$A = 600y - 2y^2$

$A' = 600 - 4y = 0$

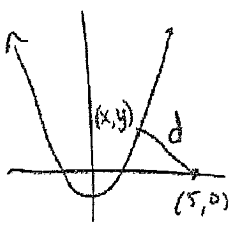
$y = 150$

max area if $y = 150$, $x = 300$

$\Rightarrow$ $\boxed{\text{Area} = 45000 \text{ ft}^2}$

A': $\oplus$ | $\ominus$ at 150

6.



Primary: $d = \sqrt{(x-5)^2 + (y-0)^2}$

$d = \sqrt{(x-5)^2 + (x^2-1)^2}$

$d = \sqrt{x^2 - 10x + 25 + x^4 - 2x^2 + 1}$

$d = \sqrt{x^4 - x^2 - 10x + 26}$

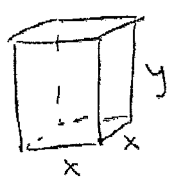
$d' = \frac{1}{2}(x^4 - x^2 - 10x + 26)^{-\frac{1}{2}}(4x^3 - 2x - 10) = \frac{2x^3 - x - 5}{\sqrt{x^4 - x^2 - 10x + 26}} = 0$

$2x^3 - x - 5 = 0 \rightarrow x = 1.4797$

d': $\ominus$ | $\oplus$ at 1.4797 (min)

$\Rightarrow$ min. distance to the point $(1.4797, 1.1895)$

7.



Primary: $SA = x^2 + 4xy$

Secondary: $x^2 y = 12$

$y = \frac{12}{x^2}$

$A = x^2 + 4x\left(\frac{12}{x^2}\right)$

$A = x^2 + 48x^{-1}$

$A' = 2x - 48x^{-2} \rightarrow A' = 2x - \frac{48}{x^2} = 0$

$2x = \frac{48}{x^2} \rightarrow 2x^3 = 48$

$x^3 = 24 \rightarrow x = \sqrt[3]{24}$

$= 2\sqrt[3]{3}$

A': $\ominus$ | $\oplus$ at $\sqrt[3]{24}$ (min.)

Min. S.A. if $x = 2\sqrt[3]{3} \approx 2.884$

$y = \frac{3}{\sqrt[3]{9}} \approx 1.442$

$SA = x^2 + 4xy$

$= 4\sqrt[3]{9} + 4(2\sqrt[3]{3})\left(\frac{3}{\sqrt[3]{9}}\right)$

$\boxed{= 4\sqrt[3]{9} + \frac{24}{\sqrt[3]{3}} \approx 24.961 \text{ ft}^2}$

8. a. $A = \pi r^2$, $r = 5.2$, $dr = \pm 0.05$

$dA = 2\pi r\,dr \Rightarrow dA = 2\pi(5.2)(\pm 0.05) = \boxed{\pm .52\pi \text{ cm}^2}$

b. $\%\text{ error} = \dfrac{dA}{A} = \dfrac{\pm .52\pi}{\pi(5.2)^2} = .01923\ldots \approx \boxed{1.9\%}$

9. a. $f'(x) = \frac{1}{2}x^{-1/2}$

Pt: $(25, 5)$

Slope: $f'(25) = \frac{1}{2}\cdot\frac{1}{5} = \frac{1}{10}$

$y - 5 = \frac{1}{10}(x - 25)$

$\boxed{y = \frac{1}{10}(x-25) + 5}$

b. $\sqrt{25.5} \approx y(25.5) = \frac{1}{10}(.5) + 5 = \frac{1}{10}\cdot\frac{1}{2} + 5 = \boxed{5\tfrac{1}{20}}$

Ch. 5

1) $\int (x^3 + 4x - 5)\,dx = \boxed{\dfrac{x^4}{4} + 2x^2 - 5x + c}$

2) $\int \sec x \tan x\,dx = \boxed{\sec x + C}$

3) $\int \cos(4x)\,dx = \frac{1}{4}\int \cos u\,du = \frac{1}{4}\sin u + C \rightarrow \boxed{\frac{1}{4}\sin(4x) + C}$

$u = 4x$

$du = 4dx \Rightarrow \frac{1}{4}du = dx$

4) $\int x(x^2-5)^{-4}\,dx = \frac{1}{2}\int u^{-4}\,du = \frac{1}{2}\cdot\frac{u^{-3}}{-3} + c \Rightarrow \boxed{-\frac{1}{6}(x^2-5)^{-3} + C}$

$u = x^2 - 5$

$du = 2x\,dx \Rightarrow \frac{1}{2}du = dx$

5) $\int \tan^3 x \sec^2 x\,dx = \int u^3\,du = \frac{u^4}{4} + c \Rightarrow \boxed{\dfrac{(\tan x)^4}{4} + C}$

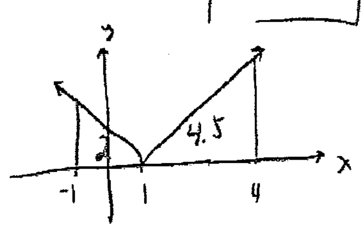
$u = \tan x$

$du = \sec^2 x\,dx$

6) $\int_0^3 (1-x)\,dx = x - \frac{x^2}{2}\Big|_0^3 = (3 - \frac{9}{2}) - (0 - 0) = \boxed{-\frac{3}{2}}$

7) $\int_{-1}^{4} |x-1|\,dx = \boxed{6.5}$

(geometric)



8) $\int_0^{\pi/3} (2\sin x + \cos x)\,dx = -2\cos x + \sin x \Big|_0^{\pi/3}$

$= \left(-2\cos\frac{\pi}{3} + \sin\frac{\pi}{3}\right) - (-2\cos 0 + \sin 0) = \left(-2\cdot\frac{1}{2} + \frac{\sqrt{3}}{2}\right) - (-2\cdot 1 + 0)$

$= -1 + \frac{\sqrt{3}}{2} + 2 \Rightarrow \boxed{1 + \frac{\sqrt{3}}{2}}$

9) $\int_0^8 (3x+1)^{1/2}\,dx \Rightarrow \frac{1}{3}\int_1^{25} u^{1/2}\,du = \frac{1}{3}\cdot\frac{2}{3}u^{3/2}\Big|_1^{25}$

$u = 3x+1$, $du = 3\,dx$, $\frac{1}{3}du = dx$; $x=0 \to u=1$, $x=8 \to u=25$

$= \frac{2}{9}\left(25^{3/2} - 1^{3/2}\right) = \frac{2}{9}(125-1) = \boxed{\frac{248}{9}}$

10. $f(c) = \frac{1}{9-1}\int_1^9 4x^{-1/2}\,dx = \frac{1}{8}\int_1^9 4x^{-1/2}\,dx = \frac{1}{8}\cdot 4\cdot 2x^{1/2}\Big|_1^9$

$= \sqrt{9} - \sqrt{1} = \boxed{2}$

$\boxed{\bigstar\ f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx}$

Ch. 6

1. a) $v(t) = 2\sin t - 1 = 0$

$\sin t = \frac{1}{2}$

$t = \frac{\pi}{6}, \frac{5\pi}{6}$

$v(t)$: $[\,(-)\ |\ (+)\ |\ (-)\,]$ on $0,\ \frac{\pi}{6},\ \frac{5\pi}{6},\ \pi$

$\boxed{\text{moving up on } (\pi/6, 5\pi/6), \text{ down on } [0,\pi/6) \cup (5\pi/6, \pi]}$

b) Displacement $= \int_0^{\pi} (2\sin t - 1)\,dt = -2\cos t - t \Big|_0^{\pi}$

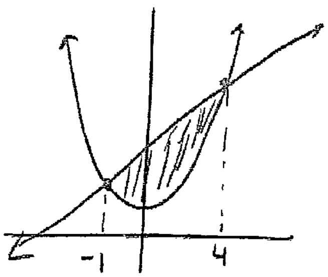
$\Rightarrow (-2\cos\pi - \pi) - (-2\cos 0 - 0)$

$= (2-\pi) - (-2) = \boxed{4-\pi \text{ ft.}}$

c) $\boxed{\text{Distance} = \int_0^{\pi} |2\sin t - 1|\,dt = 1.881}$

2. $y = x^2+1,\ y = 3x+5 \Rightarrow$

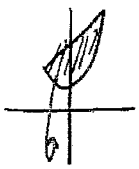
$$x^2+1 = 3x+5$$
$$x^2-3x-4=0$$
$$(x-4)(x+1)=0 \rightarrow x=4,-1$$



3. $$A = \int_{-1}^{4} \left((3x+5)-(x^2+1)\right)dx$$

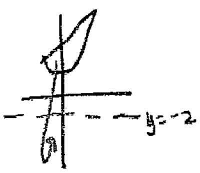
$$\boxed{A = \int_{-1}^{4} (-x^2+3x+4)\,dx}$$

4.



$$\boxed{V = \pi \int_{-1}^{4} \left[(3x+5)^2-(x^2+1)^2\right]dx}$$

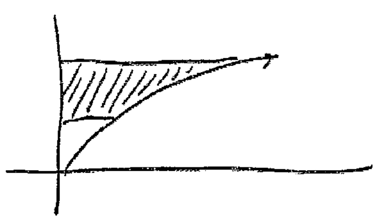
5.



$$r_o = 3x+5--2 = 3x+7$$
$$r_I = x^2+1--2 = x^2+3$$

$$\boxed{V = \pi \int_{-1}^{4} \left[(3x+7)^2-(x^2+3)^2\right]dx}$$

6. $y=\sqrt{x},\ y=1,\ y=2,\ y$-axis $\Rightarrow$ $x=y^2$



$$A = \int_1^2 (y^2)^2\,dy = \int_1^2 y^4\,dy = \frac{y^5}{5}\Big|_1^2$$

$$= \frac{32}{5} - \frac{1}{5} = \frac{31}{5}$$

Ch. 7

1) $\int (3-2x)^{-1}\,dx = -\frac{1}{2}\int u^{-1}\,du = -\frac{1}{2}\ln|u| + c$

$\Rightarrow \boxed{-\frac{1}{2}\ln|3-2x| + C}$

$u = 3-2x$

$du = -2\,dx \Rightarrow -\frac{1}{2}\,du = dx$

2) $\int \tan(5x)\,dx = \frac{1}{5}\int \tan u\,du = -\frac{1}{5}\ln|\cos u| + c$

$u = 5x$

$= \boxed{-\frac{1}{5}\ln|\cos(5x)| + c}$

$du = 5dx \Rightarrow \frac{1}{5}\,du = dx$

3) $\int_0^4 8x(x^2+1)^{-1}\,dx \Rightarrow 4\int u^{-1}\,du = 4\ln|u|$

$u = x^2+1$

$du = 2x\,dx \Rightarrow 4du = 8x\,dx$

$\Rightarrow 4\ln|x^2+1|\Big|_0^4$

$= 4(\ln 17 - \ln 1) = \boxed{4\ln 17}$

4. $y = 4e^{-3x} \Rightarrow y' = -12e^{-3x}$

5. $f(x) = e^{2x}\ln x \Rightarrow f'(x) = e^{2x}\cdot\frac{1}{x} + \ln x \cdot 2e^{2x}$

$\boxed{f'(x) = \frac{e^{2x}}{x} + 2e^{2x}\ln x}$

6. $\int 4e^{-3x}\,dx \Rightarrow -\frac{4}{3}\int e^{u}\,du = -\frac{4}{3}e^{u}+c \Rightarrow \boxed{-\frac{4}{3}e^{-3x}+C}$

$u=-3x$

$du=-3\,dx \Rightarrow -\frac{1}{3}du=dx$

7. $\int e^{x}(1+e^{x})^{-3}\,dx \Rightarrow \int u^{-3}\,du = \frac{u^{-2}}{-2}+c \Rightarrow \boxed{-\frac{1}{2}(1+e^{x})^{-2}+C}$

$u=1+e^{x}$

$du=e^{x}\,dx$

8. $\frac{d}{dx}\left(e^{3y}+e^{2x}\right)=\frac{d}{dx}(4) \Rightarrow 3e^{3y}\frac{dy}{dx}+2e^{2x}=0$

$\frac{dy}{dx}=\boxed{\frac{-2e^{2x}}{3e^{3y}}}$